ON THE MEAN-VALUE THEOREM CORRESPONDING TO A GIVEN LINEAR HOMOGENEOUS DIFFERENTIAL EQUATION *

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I consider the linear homogeneous differential expression of order n

$$Lf(x) \equiv f^{(n)}(x) + \phi_1(x)f^{(n-1)}(x) + \phi_2(x)f^{(n-2)}(x) + \cdots + \phi_n(x)f(x).$$

I suppose $\phi_1(x)$, $\phi_2(x)$, $\cdots$, $\phi_n(x)$ are continuous, $f(x)$ is differentiable n times, the variable x and all the functions in question are real. I then prove some theorems concerning the equation

$$(1) \qquad Ly \equiv \frac{d^n y}{dx^n} + \phi_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + \phi_n y = 0$$

which are, I believe, new, but there is one special case of them which belongs to the most classical part of the differential calculus. This is the case of the simplest equation of the form (1), that is

$$(2) \qquad \frac{d^n y}{dx^n} = 0.$$

Denote the Wronskian

$$\begin{vmatrix} f_1(x) & f_1'(x) & \cdots & f_1^{(n-1)}(x) \\ f_2(x) & f_2'(x) & \cdots & f_2^{(n-1)}(x) \\ \cdot & \cdot & \cdot & \cdot \\ f_n(x) & f_n'(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

by $W(f_1(x), f_2(x), \cdots, f_n(x))$. The equation (1) possesses in certain intervals the

Property W. *There exist $n - 1$ integrals $h_1(x)$, $h_2(x)$, $\cdots$, $h_{n-1}(x)$ of the homogeneous equation (1) satisfying the $n - 1$ inequalities*

$$(3) \quad h_1(x) > 0, \qquad W(h_1(x), h_2(x)) > 0,$$
$$\cdots, W(h_1(x), h_2(x), \cdots, h_{n-1}(x)) > 0$$

throughout the open interval (a, b).

Determine $n - 1$ particular integrals of (1) by such initial conditions that the $n - 1$ Wronskians in question are $= 1$ for $x = x_0$; then the inequalities (3) are satisfied in a sufficiently small interval surrounding the point x_0.

Equation (2) possesses the property W in the interval $(-\infty, +\infty)$ since the functions $1, x, x^2, \cdots, x^{n-2}$ are integrals of (2) and we have

$$1 > 0, \quad W(1,x) = 1! > 0, \quad W(1,x,x^2) = 1!\,2! > 0, \quad \cdots.$$

The equation

$$(4) \qquad\qquad\qquad\qquad y'' + y = 0$$

is of the second order. Thus there is only one inequality (3), which is satisfied by the integral $\sin(x-a)$ for $a < x < a + \pi$. No integral of the equation (4) is > 0 throughout an interval (a, b) where $b - a > \pi$. Therefore the property W belongs to (4) in any interval of length $\leqq \pi$ but in no interval of length $> \pi$. An equation $y' + \phi y = 0$ of the first order possesses the property W in any interval where ϕ is continuous.

We assume now (and also in §§ 3–5) *the existence of the property W*. Then we have the following theorems I, II, III:

THEOREM I. *If the function $f(x)$, supposed differentiable n times, vanishes at $n + 1$ points of the interval (a, b), then there exists an intermediate point ξ such that*

$$Lf(\xi) = f^{(n)}(\xi) + \phi_1(\xi)f^{(-1)}(\xi) + \cdots + \phi_n(\xi)f(\xi) = 0.$$

Theorem I is a generalization of Rolle's theorem. Apply I to the equation (2) putting $n = 1$; then we have the ordinary theorem of Rolle.

THEOREM II. *There exists one and only one integral $H(x)$ of the homogeneous equation (1) assuming n given values at n given points of the interval (a, b).*

Theorem II asserts the possibility of a certain interpolation; in the case of equation (2) and if the n points are distinct, it asserts the possibility of solving the problem which is solved explicitly by Lagrange's formula.

THEOREM III. *Determine an integral $H(x)$ of the homogeneous equation (1) assuming the same values as $f(x)$ at n given points of (a, b); determine further an integral $N(x)$ of the non-homogeneous equation*

$$(5) \qquad\qquad Ly \equiv \frac{d^n y}{dx^n} + \phi_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + \phi_n y = 1,$$

that vanishes at the n points in question. There exists a point ξ intermediate between these n points and an arbitrary point x of (a, b) such that

$$f(x) = H(x) + N(x)Lf(\xi).$$

Take the equation (2) and n coincident points; in this particular case Theorem III gives Taylor's formula with Lagrange's remainder-term.

In all the theorems just stated coinciding points are admissible; this will be fully explained in § 1. § 2 enumerates some useful formulas concerning

Wronskians. The proofs of Theorems I, II, III and a few illustrative examples are given in §§ 3–5. In § 6 I try to justify the definition of property W by proving the converse of Theorem II. § 7 deals with a system of several functions of one variable and generalizes Rolle's theorem in another manner.

1. The function $f(x)$ is supposed differentiable n times. We say that $f(x)$ vanishes at k points coinciding with x_0 if

$$(6) \qquad\qquad f(x_0) = f'(x_0) = \cdots = f^{(k-1)}(x_0) = 0.$$

This statement has a definite meaning if $k \leqq n + 1$. If we say that $f(x)$ vanishes at k *and no more* points coinciding with x_0, then we add to (6) the condition

$$f^{(k)}(x_0) \neq 0,$$

and we suppose $k \leqq n$. If $f(x)$ vanishes at

$$(7) \qquad\qquad m_1, \ m_2, \ m_3, \ \cdots, \ m_l$$

points coinciding with

$$(8) \qquad\qquad x_1, \ x_2, \ x_3, \ \cdots, \ x_l$$

respectively where

$$(9) \qquad\qquad a < x_1 < x_2 < x_3 < \cdots < x_l < b$$

then we say that $f(x)$ vanishes at $m_1 + m_2 + \cdots + m_l$ points of the interval (a, b). If we say that $f(x)$ and $\phi(x)$ assume the same values at n points of (a, b) we mean that $f(x) - \phi(x)$ vanishes at n points of (a, b). Theorem II may be stated more explicitly thus: if there are given l points (8) satisfying (9), l positive integers (7) such that $m_1 + m_2 + \cdots + m_l = n$, and finally l systems of values

$$\begin{aligned}
& y_1, \ y_1', \ y_1'', \ \cdots, \ y_1^{(m_1-1)}, \\
& y_2, \ y_2', \ y_2'', \ \cdots, \ y_2^{(m_2-1)}, \\
& \qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
& y_l, \ y_l', \ y_l'', \ \cdots, \ y_l^{(m_l-1)},
\end{aligned}$$

then there exists an integral $H(x)$ of (1) satisfying the conditions

$$(10) \quad \begin{aligned}
& H(x_1) = y_1, \quad H'(x_1) = y_1', \quad \cdots, \quad H^{(m_1-1)}(x_1) = y_1^{(m_1-1)}, \\
& H(x_2) = y_2, \quad H'(x_2) = y_2', \quad \cdots, \quad H^{(m_2-1)}(x_2) = y_2^{(m_2-1)}, \\
& \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
& H(x_l) = y_l, \quad H'(x_l) = y_l', \quad \cdots, \quad H^{(m_l-1)}(x_l) = y_l^{(m_l-1)},
\end{aligned}$$

and $H(x)$ is completely determined by these conditions.

A point ξ satisfying the inequality $x_1 < \xi < x_l$ is said to be *intermediate* between the points (8); so far we have supposed $l > 1$. If $l = 1$, that is, if all the points in question coincide with one point x_1, then $\xi = x_1$ is the only *intermediate* point.

I enumerate a few evident propositions used in what follows.   The functions $u, u_1, \cdots, u_k$ are supposed continuous and differentiable as many times as required to calculate the expression

$$
(11) \quad u_k \frac{d}{dx}\left(u_{k-1} \frac{d}{dx}\left(u_{k-2} \cdots \frac{d}{dx}\left(u_1 \frac{d}{dx}(uf)\right)\cdots\right)\right)
$$
$$
= u_k \frac{d}{dx} u_{k-1} \frac{d}{dx} u_{k-2} \cdots \frac{d}{dx} u_1 \frac{d}{dx} uf = u u_1 u_2 \cdots u_k \frac{d^k f}{dx^k} + \cdots.
$$

1. *If $f(x)$ vanishes at k (and no more) points coinciding with x_0, then $f'(x)$ vanishes at $k-1$ (and no more) points coinciding with x_0.*

2. *If $f(x)$ vanishes at k points coinciding with x_0, then $u(x)f(x)$ vanishes at k (or more) points coinciding with x_0.*

3. *Suppose $f(x)$ vanishes at k and no more points coinciding with x_0, and suppose further that $u, u_1, u_2, \cdots, u_k$ are different from zero for $x = x_0$.   Then the expression (11) is $\neq 0$ for $x = x_0$.*

4. *Suppose $f(x)$ vanishes at $k+1$ points.   Then, between these $k+1$ points, there exists an intermediate point ξ at which the expression (11) vanishes.*

We obtain 4 by repeated application of Rolle's theorem and of Theorems 1, 2.

2. Let $c_1, c_2, \cdots, c_{n-1}$ be constants; then we have

$$
(12) \quad W(f_1, f_2, \cdots, f_{n-1}, c_1 f_1 + c_2 f_2 + \cdots + c_{n-1} f_{n-1} + f_n)
$$
$$
= W(f_1, f_2, \cdots, f_{n-1}, f_n).
$$

Suppose the function u is differentiable $n-1$ times; then we have

$$
W(uf_1, uf_2, \cdots, uf_n) = u^n W(f_1, f_2, \cdots, f_n).
$$

In particular put $u = 1/f_1$; we obtain

$$
(13) \quad W\left[\left(\frac{f_2}{f_1}\right)', \left(\frac{f_3}{f_1}\right)', \cdots, \left(\frac{f_n}{f_1}\right)'\right] = \frac{W(f_1, f_2, \cdots, f_n)}{f_1^n}.
$$

For the derivative of a Wronskian we have

$$
(14) \quad W'(f_1, f_2, \cdots, f_n) = |f_k, f_k', \cdots, f_k^{(n-2)}, f_k^{(n)}|_{k=1, 2, \cdots, n}.
$$

Using (14) we obtain by the usual formula for a minor of the adjoint determinant

$$
W(f_1, \cdots, f_{n-2}, f_{n-1}) W'(f_1, \cdots, f_{n-2}, f_n)
$$
$$
- W'(f_1, \cdots, f_{n-2}, f_{n-1}) W(f_1, \cdots, f_{n-2}, f_n)
$$
$$
= W(f_1, \cdots, f_{n-2}) W(f_1, \cdots, f_{n-2}, f_{n-1}, f_n),
$$

$$
(15) \quad \frac{d}{dx} \frac{W(f_1, \cdots, f_{n-2}, f_n)}{W(f_1, \cdots, f_{n-2}, f_{n-1})}
$$
$$
= \frac{W(f_1, \cdots, f_{n-2}) W(f_1, \cdots, f_{n-2}, f_{n-1}, f_n)}{(W(f_1, \cdots, f_{n-2}, f_{n-1}))^2}.
$$

If $h_1(x)$, $h_2(x)$, $\cdots$, $h_n(x)$ are n linearly independent integrals of (1), we have

$$(16) \qquad Ly \equiv \frac{W(h_1, h_2, \cdots, h_n, y)}{W(h_1, h_2, \cdots, h_n)},$$

$$(17) \qquad W(h_1, h_2, \cdots, h_n) = ce^{-\int \phi_1(x)dx},$$

where c is a constant.   Put

$$y = Y_0, \qquad W(h_1, y) = Y_1,$$
$$W(h_1, h_2, y) = Y_2, \qquad \cdots, \qquad W(h_1, h_2, \cdots, h_n, y) = Y_n,$$
$$1 = W_0, \qquad h_1 = W_1,$$
$$W(h_1, h_2) = W_2, \qquad \cdots, \qquad W(h_1, h_2, \cdots, h_n) \quad = W_n.$$

Applying (15) we obtain

$$\frac{d}{dx}\frac{Y_0}{W_1} = \frac{W_0\, Y_1}{W_1^2}, \qquad \frac{d}{dx}\frac{Y_1}{W_2} = \frac{W_1\, Y_2}{W_2^2}, \qquad \cdots, \qquad \frac{d}{dx}\frac{Y_{n-1}}{W_n} = \frac{W_{n-1}\, Y_n}{W_n^2}.$$

Combining these formulas with (16) we find finally

$$(18) \qquad Ly \equiv \frac{W_n}{W_{n-1}}\frac{d}{dx}\frac{W_{n-1}^2}{W_{n-2}\, W_n} \cdots \frac{d}{dx}\frac{W_2^2}{W_1\, W_3}\frac{d}{dx}\frac{W_1^2}{W_0\, W_2}\frac{d}{dx}\frac{y}{W_1}.$$

(18) is the usual decomposition of a linear homogeneous differential expression into a "product" of differential expressions of the first order.   The proof here given is perhaps more direct than the usual.*

3. The property W implies the non-vanishing of $W_1, W_2, \cdots, W_{n-1}$ appearing in formula (18). W_n vanishes nowhere, by virtue of (17). Applying Theorem 4 of § 1 to the decomposition (18) we get at once Theorem I.

Apply now Theorem I to $f(x) - H(x)$ instead of $f(x)$, where $H(x)$ is an integral of the homogeneous equation (1).   Then $LH(x) = 0$ and we obtain the corollary

THEOREM I∗.   *If the function $f(x)$, supposed differentiable n times, assumes at $n + 1$ points of the interval (a, b) the same values as an integral of the homogeneous equation* (1), *there exists an intermediate point ξ such that*

$$Lf(\xi) = f^{(n)}(\xi) + \phi_1(\xi)f^{(n-1)}(\xi) + \cdots + \phi_n(\xi)f(\xi) = 0.$$

Theorem I∗ was incidentally remarked by H. Poincaré† in the case of an equation of the second order where the property W is particularly simple, including only one inequality.   Poincaré's proof is very different from the proof given here and more complicated.   I obtained the results presented in

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* Cf. L. Schlesinger, *Handbuch der Theorie der linearen Differentialgleichungen*, vol. I, p. 52, formula (14).

† L'Intermédiaire des Mathématiciens, vol. 1 (1894), pp. 141–144 and also pp. 69, 127, 172, 216.

this paper in clearing up and working out the original remark made by Poincaré.

Another corollary to Theorem I is the following

THEOREM I**. *No integral $N(x)$ of the non-homogeneous equation (5) vanishes at more than n points of the interval (a, b).*

Suppose, if possible, that $N(x)$ vanishes at $n + 1$ points; then we have by Theorem I

$$LN(\xi) = 0$$

at an intermediate point ξ; now, actually, $LN(\xi) = 1$ and thus Theorem I** is proved.

4. The general integral of (1) has the form

$$H(x) = c_1 h_1(x) + c_2 h_2(x) + \cdots + c_n h_n(x),$$

where $c_1, c_2, \cdots, c_n$ are constants. The conditions (10) form a system of n linear non-homogeneous equations with the aid of which we have to determine $c_1, c_2, \cdots, c_n$. We have to prove that this system is neither incompatible nor indeterminate. Thus we have to prove that the only solution of the *homogeneous* system obtained from (10) by putting

$$(19) \qquad\qquad y_1 = y_1' = y_2'' = \cdots = y_l^{(m_l-1)} = 0$$

is the identically vanishing solution.

Suppose, if possible, that there is a solution of the homogeneous system (10) (19) which does not vanish identically; denote it by

$$H(x) = \gamma_1 h_1(x) + \gamma_2 h_2(x) + \cdots + \gamma_k h_k(x) + \cdots + \gamma_n h_n(x)$$

where

$$\gamma_k \neq 0, \qquad \gamma_{k+1} = \gamma_{k+2} = \cdots = \gamma_n = 0.$$

Then $h_k(x)$ assumes at $n \geqq (k-1) + 1$ points the same values as an integral of the equation

$$\frac{W(h_1, h_2, \cdots, h_{k-1}, y)}{W(h_1, h_2, \cdots, h_{k-1})} = 0$$

of order $k - 1$. Then there exists, by virtue of I*, an intermediate point ξ at which

$$W(h_1(\xi), h_2(\xi), \cdots, h_{k-1}(\xi), h_k(\xi)) = 0.$$

This last equation is excluded if we assume the existence of the property W and thus Theorem II is proved.

Example. Assume that the algebraic equation

$$x^n + c_1 x^{n-1} + \cdots + c_n = 0$$

has n different real roots $a_1, a_2, \cdots, a_n$, where

$$(20) \qquad\qquad a_1 < a_2 < a_3 < \cdots < a_n.$$

Then the linear homogeneous equation with constant coefficients

$$y^{(n)} + c_1 y^{(n-1)} + c_2 y^{(n-2)} + \cdots + c_n y = 0$$

has the integrals

(21) $$e^{a_1 x}, \; e^{a_2 x}, \; \cdots, \; e^{a_n x},$$

and possesses the property W in the interval $(-\infty, +\infty)$ since the Wronskians

$$e^{a_1 x}, \; W(e^{a_1 x}, e^{a_2 x}), \; \cdots, \; W(e^{a_1 x}, e^{a_2 x}, \cdots, e^{a_n x})$$

divided by

$$e^{a_1 x}, \; e^{(a_1 + a_2) x}, \; \cdots, \; e^{(a_1 + a_2 + \cdots + a_n) x}$$

respectively reduce to positive constants which may be calculated by Vandermonde's formula. Therefore we can interpolate by a suitably chosen linear combination of the n integrals (21) n arbitrary values at the n points $x_1, x_2, \cdots, x_n$ where

(22) $$x_1 < x_2 < x_3 < \cdots < x_n.$$

This fact is equivalent to the statement that

(23)
$$\begin{vmatrix} e^{a_1 x_1} & e^{a_1 x_2} & \cdots & e^{a_1 x_n} \\ e^{a_2 x_1} & e^{a_2 x_2} & \cdots & e^{a_2 x_n} \\ \cdot & \cdot & \cdot \cdot \cdot & \cdot \\ e^{a_n x_1} & e^{a_n x_2} & \cdots & e^{a_n x_n} \end{vmatrix} \neq 0.$$

Thus we have proved (23) as a particular case of Theorem II. It may be shown directly in a slightly different manner that the determinant (23) is > 0 if (20) (22) are satisfied. I add that the property W belongs to any equation whatever with constant coefficients in the interval $(-\infty, +\infty)$ if the roots of the corresponding algebraic equation are all real, *whether distinct or not*. We get a simple proof of this from (17).

Any integral $N(x)$ of the non-homogeneous equation (5) is of the form

$$N(x) = N_0(x) + H(x).$$

Here $N_0(x)$ is a particular integral of the non-homogeneous equation (5) and $H(x)$ is the general integral of the homogeneous equation (1); $H(x)$ contains n arbitrary constants which we can adapt to any conditions of the form (10), by virtue of Theorem II. Thus we obtain the corollary

THEOREM II∗. *There exists one and only one integral $N(x)$ of the non-homogeneous equation (5) assuming n given values at n given points of the interval (a, b).*

5. I now prove Theorem III. The possibility of determining $H(x)$ according to the enunciated conditions follows from Theorem II and the possibility

of determining $N(x)$ follows from II*. Let x_0 be a point of (a, b) but not one of the n points mentioned at which $f(x) - H(x)$ and $N(x)$ vanish. Then we have, by Theorem I**, $N(x_0) \neq 0$. Therefore it is possible to find a constant C satisfying the equation

(24) $$f(x_0) = H(x_0) + CN(x_0).$$

Thus the function

$$f(x) - H(x) - CN(x)$$

vanishes at $n + 1$ points of (a, b). There exists, by Theorem I, an intermediate point ξ at which

$$L(f(\xi) - H(\xi) - CN(\xi)) = Lf(\xi) - 0 - C \cdot 1 = 0,$$

that is,

$$C = Lf(\xi).$$

Substituting this value in (24) we have Theorem III.*

I will now give some concrete cases of Theorem III with immediate applications.

Example 1. Let x_1 be the point at which we interpolate the differentiable function $f(x)$ by an integral of the equation

$$y' + y = 0.$$

Adopting the notation of Theorem III we have

$$H(x) = f(x_1) e^{x_1 - x}, \qquad N(x) = 1 - e^{x_1 - x},$$

and consequently for $x > x_1$

(25) $$f(x) = f(x_1) e^{x_1 - x} + (1 - e^{x_1 - x})(f(\xi) + f'(\xi))$$

where $x_1 < \xi < x$. From (25) we get at once the proposition †

Suppose $f(x)$ differentiable for $x > 0$; then

$$\lim_{x \to +\infty} (f(x) + f'(x)) = c$$

implies $\lim f(x) = c$.

We may assume $c = 0$ (and apply the proof to $f(x) - c$ instead of to $f(x)$). Choose first x_1 sufficiently great so that $|f(x) + f'(x)| < \epsilon/2$ for $x > x_1$; then choose x_2, $x_2 > x_1$, such that $|f(x_1)| e^{x_1 - x_2} < \epsilon/2$. Finally we get by (25) for $x > x_2$

$$|f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} .$$

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* The particular case corresponding to (2) was given by Ch. Hermite for points coinciding in an arbitrary manner. See *Œuvres*, vol. III, pp. 432–443, and T. J. Stieltjes, *Œuvres*, vol. I, pp. 47–60. For a more general point of view see G. D. Birkhoff, these T r a n s a c t i o n s, vol. 7 (1906), pp. 107–136.

† G. H. Hardy, Q u a r t e r l y   J o u r n a l   o f   M a t h e m a t i c s, vol. 35 (1903), pp. 31–33. The proof given by O. Perron, M a t h e m a t i s c h e   Z e i t s c h r i f t, vol. 6 (1920), p. 159, supposes *more* than the derivability of $f(x)$.

By the same argument we may prove an analogous result for any equation with constant coefficients if the roots of the corresponding algebraic equation are real and negative.[*]

Example 2.   In dealing with the equation

$$(4) \qquad y'' + y = 0$$

we may choose for (a, b) any interval of length π.   The two points at which we interpolate may be different or not.   Interpolating at two distinct points x_1, x_2, where

$$(26) \qquad x_1 < x_2 < x_1 + \pi,$$

we find the formula

$$
(27) \qquad
\begin{aligned}
f(x) &= \frac{f(x_1)\sin(x_2 - x) + f(x_2)\sin(x - x_1)}{\sin(x_2 - x_1)} \\
&+ \left(1 - \frac{\cos\left(x - \dfrac{x_1 + x_2}{2}\right)}{\cos\dfrac{x_2 - x_1}{2}}\right)(f''(\xi) + f(\xi))
\end{aligned}
$$

valid for $x_2 - \pi < x < x_1 + \pi$; ξ is an interior point of the least interval containing x_1, x_2, x.   Interpolating at two points coinciding with x_1 we find

$$
(28) \qquad
\begin{aligned}
f(x) &= f(x_1)\cos(x - x_1) + f'(x_1)\sin(x - x_1) \\
&+ (1 - \cos(x - x_1))(f''(\xi) + f(\xi))
\end{aligned}
$$

valid for $x_1 - \pi < x < x_1 + \pi$; ξ lies between x_1 and x.   We see that restriction (26) is essential for the validity of (27), $x_2 = x_1 + \pi$ being generally inadmissible; this point will be cleared up by Theorem IV.

In order to give an application suppose $f(x)$ is an integral of the equation

$$(29) \qquad y'' + \phi(x)y = 0,$$

where

$$(30) \qquad \phi(x) > 1;$$

suppose further that $f(x_1) > 0$.   We have by (29)

$$f''(\xi) + f(\xi) = -(\phi(\xi) - 1)f(\xi);$$

thus we may put (28) in the form

$$(31) \quad f(x) = A\sin(x - a) - (1 - \cos(x - x_1))(\phi(\xi) - 1)f(\xi)$$

[*] O. Perron, Mathematische Zeitschrift, vol. 6 (1920), pp. 161–163.

where

$$(32) \qquad A \sin (x_1 - a) = f(x_1) > 0, \qquad A \cos (x_1 - a) = f'(x_1).$$

The situation shown by (30) (31) (32) is this: the curve $A \sin (x - a)$ meets the curve $f(x)$ at the point x_1, see (32); the curve $A \sin (x - a)$ runs on the upper side of the curve $f(x)$ *as far* as $f(x)$ runs on the upper side of the x-axis, see (31) (30); but the curve $A \sin (x - a)$ cuts the x-axis at two points between which x_1 lies and whose distance apart is π; hence $f(x)$ reaches the x-axis necessarily *before* $A \sin (x - a)$ *cuts it* and we have the theorem

The distance between two consecutive zeros of any integral of the equation (29) *satisfying the condition* (30) *is* $< \pi$.

This is a particular case of a well known classical theorem given by Sturm[*] that can be proved generally by the argument I have used in the foregoing concrete case.

Any integral $f(x)$ of the equation (29) has the following property: the inequality

$$(33) \qquad\qquad\qquad f(x) > 0$$

implies

$$(34) \qquad\qquad\qquad f''(x) + f(x) < 0.$$

Consider the curve of which the equation is

$$(35) \qquad\qquad\qquad r = \frac{1}{f(\vartheta)}$$

in a system of polar coördinates r, ϑ with origin O. The inequality (34) expresses that the curve (35) appears from O to be *convex*, as we may see, e.g., by (27). The result just proved may be formulated geometrically as follows:

A curve appearing from a given point O to be everywhere convex subtends an angle $< \pi$ *at O.*

H. Poincaré was led to the remark above mentioned (§ 3) by constructing an analytical demonstration of this geometrically evident fact.

6. The converse of Theorem II is also true: if the interpolation of any n values by an integral of (1) is possible at any n points of an interval, then the equation (1) possesses the property W in that interval. The possibility of interpolation, that is, the *existence* of a solution of the non-homogeneous system (10) with arbitrary $y_1, y_1', \cdots, y_l^{(m_l-1)}$, is equivalent to the *non-existence* of a non-vanishing solution of the corresponding homogeneous system; we used this equivalence in § 4. Thus the most simple form of the converse to Theorem II is the following:

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[*] Cf. Maxime Bôcher, *Méthodes de Sturm*, Paris, 1917, p. 52.

THEOREM IV. *If, except the identically vanishing integral, there exists no integral of equation* (1) *vanishing at n points of the interval* $a \leqq x < b$, *then there exist* $n - 1$ *integrals* $h_1(x)$, $h_2(x)$, $\cdots$, $h_{n-1}(x)$ *such that the Wronskians*

(36) $h_1(x)$, $W(h_1(x), h_2(x))$, $\cdots$, $W(h_1(x), h_2(x), \cdots, h_{n-1}(x))$

vanish nowhere in the interval $a < x < b$.

The signs of the Wronskians are of no importance provided that they are constant. If the Wronskians (36) do not vanish we can make them positive by substituting $- h_\nu(x)$ for $h_\nu(x)$ for certain values of ν.

Determine n integrals $h_1(x)$, $h_2(x)$, $\cdots$, $h_n(x)$ satisfying the initial conditions

$$h_1(a) = h_1'(a) = h_1''(a) = \cdots = h_1^{(n-3)}(a) = h_1^{(n-2)}(a) = 0,$$
$$h_1^{(n-1)}(a) = 1,$$

(37) $h_2(a) = h_2'(a) = h_2''(a) = \cdots = h_2^{(n-3)}(a) = 0$, $h_2^{(n-2)}(a) = 1$,

$\quad h_{n-2}(a) = h_{n-2}'(a) = 0$, $\quad h_{n-2}''(a) = 1$,

$\quad h_{n-1}(a) = 0$, $\quad h_{n-1}'(a) = 1$,

$\quad h_n(a) = 1$.

These integrals are linearly independent because their Wronskian $= \pm 1$ for $x = a$; $h_1(x)$ vanishes at $n - 1$ points coinciding with a; by hypothesis there exists no nth point in the interval $a \leqq x < b$ at which $h_1(x)$ vanishes; therefore $h_1(x) \neq 0$ for $a < x < b$. Adopt the notation

$$W(h_1, h_2, \cdots, h_k) = W_k$$

that we used in § 2 and suppose that we have already proved

(38) $W_1 \neq 0$, $\quad W_2 \neq 0$, $\quad \cdots$, $\quad W_k \neq 0$

for $a < x < b$, $k < n$.

Let $c_1, c_2, \cdots, c_k$ be constants and put

$$c_1 h_1(x) + c_2 h_2(x) + \cdots + c_k h_k(x) + h_{k+1}(x) = h(x).$$

$h(x)$ does not vanish identically, and therefore it vanishes at no more than $n - 1$ points contained in the interval $a \leqq x < b$. Now $h(x)$ vanishes at $n - k - 1$ points coinciding with a by virtue of the conditions (37). Determine the constants $c_1, c_2, \cdots, c_k$ in such a manner that $h(x)$ vanishes at k points coinciding with x_0 where $a < x_0 < b$; this is possible because the determinant of the k equations in question

$$W(h_1(x_0), h_2(x_0), \cdots, h_k(x_0)) \neq 0,$$

by the hypothesis of complete induction, that is, by (38). Beyond the

$$(n - k - 1) + k = n - 1$$

zeros already mentioned, $h(x)$ has no other zeros for $a \leqq x < b$. In particular $h(x)$ *vanishes at no more than k points coinciding with x_0.* Consequently, by virtue of hypothesis (38) and of the Theorem 3 of § 1,

$$(39) \quad \frac{W_k^2}{W_{k-1}} \frac{d}{dx} \frac{W_{k-1}^2}{W_{k-2}\,W_k} \cdots \frac{d}{dx} \frac{W_2^2}{W_1\,W_3} \frac{d}{dx} \frac{W_1^2}{W_0\,W_2} \frac{d}{dx} \frac{h}{W_1}$$
$$= W(h_1, h_2, \cdots, h_k, h) = W(h_1, h_2, \cdots, h_k, h_{k+1}) = W_{k+1}$$

does not vanish for $x = x_0$; we obtain (39) from (15) (12) in the same manner as we obtained (18). The point x_0 was arbitrarily chosen in the interval $a < x < b$ and thus Theorem IV is proved.

7. Another demonstration of the Theorems I, II, III essentially different from that given in §§ 1–5 is based on the following theorem,

THEOREM V. *Suppose all the functions $h_1(x)$, $h_2(x)$, $\cdots$, $h_n(x)$, $f(x)$ possess derivatives of order n throughout the interval $a < x < b$ and satisfy the following n inequalities*:

$$h_1(x) > 0, \quad W(h_1(x), h_2(x)) > 0, \quad \cdots,$$
$$W(h_1(x), h_2(x), \cdots, h_n(x)) > 0.$$

Suppose also that the l points $x_1, x_2, \cdots, x_l$ and the corresponding positive integers $m_1, m_2, \cdots, m_l$ satisfy the conditions

$$a < x_1 < x_2 < \cdots < x_l < b, \quad m_1 + m_2 + \cdots + m_l = n + 1.$$

Consider the determinant of order $n + 1$

$$(40) \quad \left| h_k(x_1), h_k'(x_1), \cdots, h_k^{(m_1-1)}(x_1), h_k(x_2), h_k'(x_2), \right.$$
$$\left. \cdots, h_k^{(m_2-1)}(x_2), \cdots, h_k(x_l), \cdots, h_k^{(m_l-1)}(x_l) \right|,$$

the n first rows being obtained by putting successively $k = 1, 2, 3, \cdots, n$ and the last row by substituting $f(x)$ for $h_k(x)$.

There exists an intermediate point ξ, $x_1 < \xi < x_l$, such that the value assumed by the Wronskian at the point ξ, that is,

$$W(h_1(\xi), h_2(\xi), \cdots, h_n(\xi), f(\xi)),$$

is $>$, $=$ or < 0, according as the determinant (40) *is $>$, $=$ or < 0.*

Theorem V may be compared to a well known determinantal theorem of H. A. Schwarz.* Theorem V may be proved by complete induction; its validity for the $(n - 1) + 1$ functions

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* H. A. Schwarz, *Abhandlungen*, vol. II, pp. 296–302; T. J. Stieltjes, *Œuvres*, vol. II, pp. 110–123. The determinant (40) has been considered by H. M. Morse, G. A. Pfeiffer, and G. M. Green from a different point of view; see Bulletin of the American Mathematical Society, vol. 23 (1916), pp. 114–122.

$$\left(\frac{h_2\,(x)}{h_1\,(x)}\right)',\,\left(\frac{h_3\,(x)}{h_1\,(x)}\right)',\,\cdots,\left(\frac{h_n\,(x)}{h_1\,(x)}\right)',\left(\frac{f\,(x)}{h_1\,(x)}\right)'$$

implies its validity for the $n+1$ functions $h_1\,(x),\,h_2\,(x),\,\cdots,\,h_n\,(x),\,f(x)$, as may be shown by (13) and by Rolle's theorem. I had originally based my demonstration of Theorems I, II, III on Theorem V. I was led to the treatment of the subject I finally adopted by a kind remark made by Professor H. Weyl.

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ERRATA, VOLUME 24

J. F. RITT, *On algebraic functions which can be expressed in terms of radicals.* Page 21, lines 30 and 33, for "n^2" read "n".